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Non-perturbative renormalization of HQET and QCD*

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We discuss the necessity of non-perturbative renormalization in QCD and HQET and explain the general strategy for solving this problem. A few selected topics are discussed in some detail, namely the importance of off-shell improvement in the MOM-scheme on the lattice, recent progress in the implementation of finite volume schemes and then particular emphasis is put on the recent idea to carry out a non-perturbative renormalization of the *Heavy Quark Effective Theory* (HQET).

1. INTRODUCTION

Non-perturbative renormalization is an important mile-stone on the path from lattice QCD computations to precise predictions for particle physics phenomenology. Why is this so? One might think that only the QCD coupling and quark masses need to be renormalized. However, very important phenomenological applications of lattice QCD concern the physics of heavy quarks and weak decays. As illustrated in Fig. 1, these involve energy scales (masses), which are far too high for the lattices that can be simulated on present and (near future) computers. Effective theories have to be used, summarizing the effects of the heavy fields, which can't be treated dynamically, in local composite operators. They are a way to implement expansions in terms of $1/M_x$, $x = \mathrm{W}, \mathrm{t}, \ldots$. The requirement that the corrections to the n-th order are really $(1/M_x)^{n+1}$ and terms such as $\alpha_\mathrm{s}/(M_x)^n$ do not appear fixes the renormalization of the operators in the effective Lagrangian. We denote the so-defined renormalization scheme by "match".

To give a specific example, the mixing amplitude between a K-meson and a $\overline{\mathrm{K}}$ is to lowest order in the weak interaction given by a QCD matrix element of a 4-fermion operator whose (re)normalization is completely fixed by the matching to the standard model amplitude.

Of course, already before considering weak decays, the renormalization of the QCD coupling and quark masses is important to obtain the basic renormalization group invariants (RGI) of the theory: the Lambda parameter, $\Lambda_{\overline{\mathrm{MS}}}$, and the RGI quark masses, $M_i, i = \mathrm{u}, \mathrm{d}, \ldots$. Their definition and their special rôle in parameterizing QCD is briefly but thoroughly explained in [1].

Why should we do renormalization non-perturbatively? There are good reasons:

1. Concerning $\Lambda_{\overline{\mathrm{MS}}}$, M_i, phenomenological determinations claim very good precision but are frequently based on difficult to test assumptions, most notably the applicability of perturbation theory in various kinematical regions. In my opinion, the rôle of lattice QCD is complementary: to use as few assumptions as possible. This means to perform renormalization *non*-perturbatively.

2. Some examples demonstrate that perturbative renormalization is not sufficiently precise in practice. E.g. the JLQCD collaboration computed B_K with two different perturbatively renormalized operators which should be equivalent in the continuum limit. Performing the continuum extrapolations, they could only find agreement by fitting α^2 corrections in addition to the a^2 lattice artifacts [2]. A more recent example is the static-light axial current. As shown in Fig. 2 perturbative and non-perturbative Z-factors differ by more than the unavoidable $\mathrm{O}(a)$ term.

2. GENERAL STRATEGY

We now discuss the implementation of scale-dependent renormalizations, leaving aside simpler cases such as the renormalization of the axial current [6], where the (scale independent) renormal-

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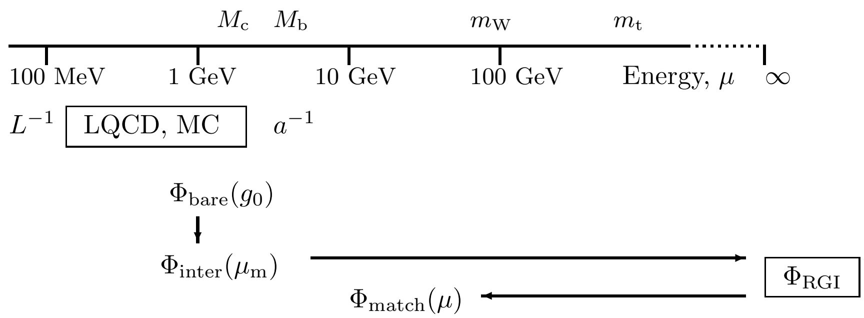


Figure 1. Top: The standard model summarizes physics at many different scales, while lattice QCD simulated by Monte Carlo (MC) can correctly cover only a small range between the inverse lattice spacing a^{-1}, and the box size L. Bottom: The strategy to connect the bare matrix elements to the ones in the matching scheme.

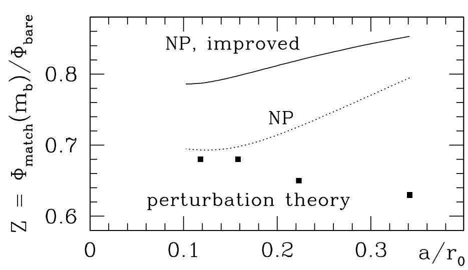


Figure 2. Non-perturbative (NP) renormalization constant of the static-light axial current [3] and its estimate[4] from (tadpole-improved [5]) perturbation theory. The label "improved" refers to the O(a)-improved theory.

ization may be determined from a symmetry. Focusing on multiplicative renormalization, the bare operators, $\mathcal{O}_{\rm bare}$, and the renormalized ones at scale μ, $\mathcal{O}_{\rm inter}(\mu)$, are related by

$$\mathcal{O}_{\rm inter}(\mu) = Z_{\rm inter}(g_0, a\mu)\mathcal{O}_{\rm bare}\,, \tag{1}$$

with a renormalization factor Z (or a matrix) depending on the bare coupling, g_0, and the combination $a\mu$. In an *intermediate renormalization scheme*, Z is fixed by requiring

$$\langle\beta|\mathcal{O}_{\rm inter}(\mu)|\alpha\rangle = \langle\beta|\mathcal{O}_{\rm bare}|\alpha\rangle_{\rm treelevel} \tag{2}$$

for convenient states $|\alpha\rangle, |\beta\rangle$. To have simple renormalization group equations (RGE), it is important that $|\alpha\rangle, |\beta\rangle$ are characterized through one scale, μ, only. In the two most frequently used intermediate schemes, this is realized as follows.

SF: In the Schrödinger functional schemes [7], eq. (2) is formulated through the gauge invariant path integral in a finite space-time volume $T \times L \times L \times L$ with T/L fixed. $|\alpha\rangle, |\beta\rangle$ are given in terms of some boundary states (propagated in Euclidean time) and one has $\mu = 1/L$.

MOM [8]: Here one uses infinite volume quark states in Landau gauge, with four-momenta p of the quarks satisfying $p^2 = \mu^2$.

Both are massless schemes, their renormalization constants are evaluated at zero quark masses. In addition, the renormalized operators are independent of the regularization.[2] More precisely, (the limit $a \to 0$ of) the matrix elements

$$\Phi_{\rm inter}(\mu) \equiv \langle f|\mathcal{O}_{\rm inter}(\mu)|i\rangle \tag{3}$$

are independent of the regularization and the perturbative coefficients γ_i in the RGE

$$\mu {\rm d}\Phi_{\rm inter}(\mu){\rm d}\mu \;=\; \gamma(\bar g^2(\mu))\,\Phi_{\rm inter}(\mu) \tag{4}$$

$$\gamma(\bar g^2) \;=\; -[\gamma_0 \bar g^2 + \gamma_1 \bar g^4 + \ldots] \tag{5}$$

can in principle be computed using dimensional or a lattice regularization.

For phenomenological applications we are interested in the matrix elements $\Phi_{\rm match}(\mu)$ in the matching scheme mentioned in Sect. 1. To connect to it, it is usually[3] convenient to first compute the *scheme independent* renormalization

[2] Thus also the name RI/MOM is used instead of MOM.
[3] For an alternative see Sect. 3.3.

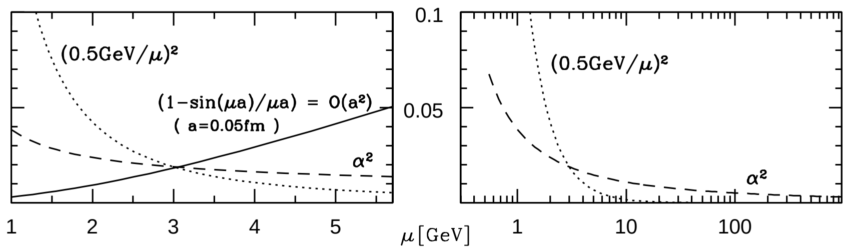


Figure 3. Typical relative error terms as they show up in infinite volume schemes (left) and finite volume schemes (right): a^2-effects as well as non-perturbative (dotted) and perturbative (dashed) terms.

group invariant (RGI)

$$\Phi_{\rm RGI} = \lim_{\mu\to\infty} \Phi_{\rm inter}(\mu) \left[2b_0\bar{g}^2(\mu)\right]^{-\gamma_0/2b_0^2} \quad (6)$$

$$= \Phi_{\rm inter}(\mu)(2b_0\bar{g}(\mu)^2)^{-\gamma_0/2b_0^2} \quad (7)$$
$$\times \exp\left\{-\int_0^{\bar{g}(\mu)} {\rm d}g[\gamma(g)\beta(g) - \gamma_0 b_0 g]\right\}$$

and then connect $\Phi_{\rm RGI}$ to $\Phi_{\rm match}(\mu)$ through the analogue of eqs.(4,6) for the matching scheme. This last step is mostly done by perturbation theory and – as discussed by P. Hasenfratz at last year's conference [9] – its reliability needs to be investigated. For the purpose of this report, we note that the RGI matrix elements eq. (6) are the fundamental quantities of QCD and should be determined from lattice QCD with good precision.

The strategy is sketched in the bottom part of Fig. 1. For multiplicative renormalization, each arrow in this graph is realized by a factor,

$$\Phi_{\rm match}(\mu) = \Phi_{\rm match}(\mu)\Phi_{\rm RGI} \times \Phi_{\rm RGI}\Phi_{\rm inter}(\mu_{\rm m}) \quad (8)$$
$$Z_{\rm inter}(g_0,\mu_{\rm m}a) \times \Phi_{\rm bare}(g_0)\,.$$

The first two factors are independent of the lattice regularization (e.g. choice of action).

2.1. Challenges

The practical implementation of this programme represents a challenge beyond the one present in spectrum calculations, since the limit $\mu \to \infty$ in eq. (6) has to be controlled.

With MOM (or other infinite volume schemes) as an intermediate scheme, one computes $\Phi_{\rm inter}(\mu)$ with $Z_{\rm inter}(g_0, a\mu)$ from eqs.(1,2) for $\mu = {\rm O}(2\,{\rm GeV})$ and uses the *perturbative* RG eq. (5) inserted into eq. (7) to continue to $\mu \to \infty$. Systematic errors of the order shown on the l.h.s. of Fig. 3 have to be expected. It is evidently not easy to disentangle the various error sources in the window of μ and a available on large volume lattices.

To cleanly separate the lattice artifacts (O(a^2) or O(a)) from the physical μ-dependence, renormalization conditions may also be posed in finite volume (e.g. SF-scheme) and then a recursion $\mu \to 2\mu$ allows to reach large enough μ to make sure that the terms illustrated on the r.h.s. of Fig. 3 are negligible/controlled [10,11]. The key point is that before comparing the μ-dependence to the perturbative one, the a-effects can be controlled in each step of the recursion. Examples are given in Sect. 3.2.

3. SELECTED TOPICS

We now discuss some selected topics of relevance. Recent reviews, with partly different emphasis, are [12].

3.1. MOM-scheme

The MOM-scheme [8] is very popular. It is rather easy to implement and does not require simulations to be done specifically for the renormalization. A number of new numerical results have been presented at this conference [13–15]. Here we concentrate on two aspects which have not been discussed much.

3.1.1. O(a)-improvement

Renormalization conditions in the MOM-scheme are based on off-shell quark Greens-functions in Landau gauge, which *are not* O(a)-improved by the standard on-shell improvement programme [16,17]. The most basic ingredient is the momentum space quark propagator

$$\tilde{S}(p) = \sum_x \mathrm{e}^{-ipx} \langle \psi(x)\overline{\psi}(0)\rangle \tag{9}$$

$$\equiv [i\gamma_\mu p_\mu \Sigma_1 + \Sigma_2]^{-1} . \tag{10}$$

For sufficiently large p^2, it is computable in perturbation theory and Σ_2 is expected to go to a constant (asymptotic freedom!). Evaluating Σ_2 with the non-perturbatively on-shell improved action [17] (e.g. at $a = 0.07\,\mathrm{fm}$), one finds instead a strong rise with p^2 [18,19] (• in Fig. 4). The SPQ$_{\mathrm{CD}}$R Collaboration verified that this lattice artifact is reduced for decreasing a [14]. We learn that off-shell O(a) lattice artifacts are large and their impact on the numerical results in the MOM-scheme may be substantial.

The possibilities of reducing them in a systematic Symanzik improvement programme were studied in [20,21]. A rough summary follows.

Since one is interested in non gauge invariant Greens functions, the improvement terms have to satisfy only BRST invariance instead of gauge invariance. In addition, contact terms have to be subtracted from the correlation functions in position space.

We restrict ourselves to the quark propagator and correlation functions of the form $\langle\psi(x)\overline{\psi}(y)\mathcal{O}_\Gamma(z)\rangle$, with $\mathcal{O}_\Gamma = \overline{\psi}\Gamma\tau^a\psi$, Γ a matrix in Dirac-space and a traceless matrix τ^a in flavor space. These are the Green's functions needed for the MOM-renormalization of flavor non-singlet 2-fermion operators. Then off-shell O(a)-improvement may be achieved by [20]

1) replacing the quark field $\psi(x)$ with

$$\psi_{\mathrm{R}}(x) = Z_{\mathrm{q}}^{-1/2}\,(1 - 12 b_{\mathrm{q}} a m_{\mathrm{q}})\,\psi_{\mathrm{I}}(x) \tag{11}$$

$$\psi_{\mathrm{I}}(x) = [1 + a c'_{\mathrm{q}} D + a c_{\mathrm{NGI}} \partial_\mu\gamma_\mu]\psi(x) ,$$

with g_0-dependent coefficients $Z_{\mathrm{q}}, b_{\mathrm{q}}, c'_{\mathrm{q}}, c_{\mathrm{NGI}}$ (and similarly for $\overline{\psi}$).

2) adding a term $a c'_\Gamma \overline{\psi}\Gamma D\psi$ to $\mathcal{O}_\Gamma$ in addition to the on-shell improvement terms of [16].

Above, D denotes the full lattice Dirac operator including the mass term. An important point to note is that this is strictly speaking *not an improvement of the fields*: e.g. an N-point function of $\mathcal{O}_\Gamma(x)$ will be O(a)-improved when considered on-shell (all operators separated by a physical distance) but not when considered off-shell (e.g. Fourier transformed). In the latter case, additional contact terms have to be subtracted in general. Nevertheless, the above is a possible form of writing the O(a) counter-terms needed for the MOM-scheme renormalization of $\mathcal{O}_\Gamma$ [20].

Investigations of practical ways to determine the new coefficients $c'_{\mathrm{q}}, c_{\mathrm{NGI}}$ have started [19,22]. Before we discuss the result, we turn to off-shell improvement in the context of formulations of QCD with exact chiral symmetry (see [23]). Their massless Dirac operator D satisfies the Ginsparg-Wilson relation [24]

$$\gamma_5 D^{-1} + D^{-1}\gamma_5 = a\,2R\,\gamma_5 , \tag{12}$$

with a local operator R and the fermionic action is $S = a^4 \sum_x \overline{\psi}(x)[D + m_0(1 - a2RD)]\psi(x)$. Examples of *local Dirac operators* satisfying eq. (12) are known [25,26] and in addition it has been shown that Domain wall (DW) fermions [27] are approximate realizations [28].

A possibly confusing point is that the MOM-scheme is applied for these formulations, stating that there is O(a)-improvement because of chiral symmetry, while Lüscher's exact chiral symmetry [29] is only valid on-shell (otherwise it would be in contradiction to the Nielsen-Ninomia theorem [30]). We briefly explain in which sense off-shell O(a)-improvement is implied by eq. (12); see [31] for a similar discussion. For simplicity we assume that R is proportional to unity.

Introduce quark fields and anti-quark fields

$$\psi_{\mathrm{I}}(x) = (1 - a2RD)\psi(x) , \quad \overline{\psi}_{\mathrm{I}}(x) = \overline{\psi}(x) , \tag{13}$$

and perform the same replacement in quark bilinears,

$$\mathcal{O}_{\Gamma,\mathrm{I}}(x) = \overline{\psi}_{\mathrm{I}}(x)\Gamma\tau^a\psi_{\mathrm{I}}(x) . \tag{14}$$

It follows immediately that the quark propagator

$$\tilde{S}_{\mathrm{I}}(p) = \sum_x \mathrm{e}^{-ipx}\langle\psi_{\mathrm{I}}(x)\overline{\psi}_{\mathrm{I}}(0)\rangle \tag{15}$$

satisfies $\tilde{S}_{\mathrm{I}}(p)\gamma_5 + \gamma_5\tilde{S}_{\mathrm{I}}(p) = 0$ for $m_0 = 0$; the continuum relation is exact at finite lattice spacing. Furthermore, the Green's functions of quark

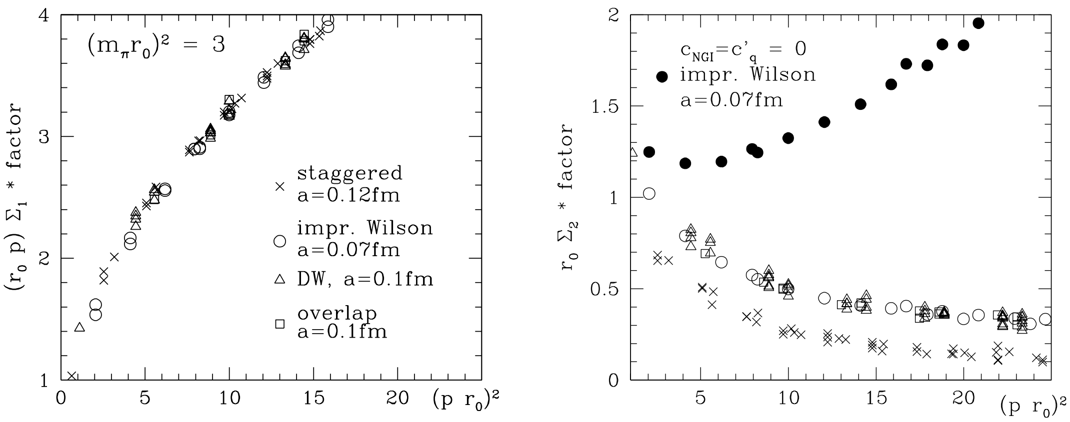


Figure 4. The functions Σ_i, eq. (10), renormalized such that they coincide at $\Sigma_1(r_0^2 p^2 = 5)$. Results are from [33] (overlap), [32] (DW), [19] (improved Wilson) and [34] (staggered). Open circles use estimates of $c'_\mathrm{q}, c_\mathrm{NGI}$ [19], while filled ones are for $c'_\mathrm{q} = c_\mathrm{NGI} = 0$. Error bars are comparable to symbol sizes.

bilinears, eq. (14), satisfy the continuum chiral Ward identities at finite lattice spacing. This excludes the necessity of additional $\mathrm{O}(a)$ improvement terms since all possible terms [20] transform differently under chiral rotations and would lead to violations of the Ward identities. One concludes that the off-shell improved quark fields and quark bilinears are known without any free improvement coefficients.

Note that the action is *not* written in terms of the fields ψ_I, $\overline{\psi}_\mathrm{I}$ since this would lead to a non-local Dirac operator (in agreement with the Nielsen-Ninomia theorem); only the fields in the correlation functions are improved. Concerning DW fermions in the limit of large "fifth dimension", the situation was clarified by Kikukawa and Noguchi. They derived a 4-d effective action with fields $\psi, \overline{\psi}$ and a local Dirac operator which satisfies eq. (12) with $R = 2$. The Shamir-Furman fermion fields[27], which are constructed from the 5-d fields at the boundaries, are identified with ψ_I, $\overline{\psi}_\mathrm{I}$ exactly related to $\psi, \overline{\psi}$ by eq. (13).

We now come back to numerical results. In Fig. 4, Σ_1 and Σ_2 are displayed for various discretizations. The results for DW fermions [32] and overlap fermions [33] show the expected leveling off of Σ_2 with p^2 and are in complete agreement with each other within their (not so small) statistical errors. As discussed before, $\mathrm{O}(a)$-errors are expected to be absent[4]. The mutual agreement is an indication that $\mathrm{O}(a^2)$-errors are small. More surprisingly, also the approximately off-shell $\mathrm{O}(a)$-improved Wilson fermion results [19] fit in very well! On the other hand, the staggered fermion Σ_2 [34] appears to have significant discretization errors at $a = 0.12$ fm.

What does this mean for NP renormalization in the MOM-scheme? Since the renormalization conditions, eq. (2), are imposed in such a way that $Z = 1 + \mathrm{O}(g_0^2)$ holds exactly at finite a, discretization errors are suppressed by one order of g_0^2. Whether this is sufficient needs more investigations. In any case, formulations with exact chiral symmetry [32,33] are expected to be superior and first numerical results for MOM-scheme Z's [32,33] appear to confirm this.

At this conference, new results for MOM renormalization constants in the on-shell improved theory were presented [13,14]. Not all of them are well compatible with the perturbative RG to the NL order. It remains to be clarified why this is so. Apart from a true non-perturbative μ-dependence, $\mathrm{O}(a)$ errors or Goldstone pole contributions are possible explanations.

[4]We assume that the extent of the fifth dimension in the DW computation was large enough.

3.1.2. Goldstone poles

To discuss the latter possibility, we choose the simplest case, $\mathcal{O}_5 = \overline{\psi}\gamma_5\tau^a\psi$ and write schematically

$$R(m) = \langle p'|\mathcal{O}_5|p\rangle \;/\; \langle p'|\mathcal{O}_5|p\rangle_{\rm treelevel}\,, \tag{16}$$

with $\langle p'|$ ($|p\rangle$) an (anti-) quark state. It is standard to choose the forward kinematics $p = p'$ and $Z_{\rm P}^{-1}$ in the MOM-scheme is to be obtained from $R(m)$ in the limit where the quark mass m goes to zero. However, it is well known (see e.g. [8,35]) that $R(m)$ is singular in the chiral limit due to the coupling of the pion to the pseudo-scalar vertex and the fact that $p - p' = 0$ in the chosen kinematics; one expects

$$R(m) = A(p^2) 1 m_\pi^2(m) + B(p^2) + \mathrm{O}(m_\pi^2)\,. \tag{17}$$

Based on this form it has become practice [13,14] to eliminate the first term by forming

$$B(p^2) \;=\; m_\pi^2(m_1)R(m_1) - m_\pi^2(m_2)R(m_2) m_\pi^2(m_1) - m_\pi^2(m_2) + \mathrm{O}(m_\pi^2)\,, \tag{18}$$

and then to determine $Z_{\rm P}^{-1}$ from the chiral limit of eq. (18). From a theoretical point of view, eq. (17) means that the renormalization constant $Z_{\rm P}$ (and others) do not exist in the massless MOM-scheme for $p = p'$. As alternatives, one could formalize the subtraction eq. (18) and include it into the *definition* of the renormalization scheme or one could choose a kinematics which truly corresponds to short distances in position space, i.e. $p^2, (p')^2, (p-p')^2 \gg \Lambda_{\rm QCD}^2$ or one could avoid operators which couple to the pion and relate their renormalization to others by Ward identities[36]. The second alternative would certainly be a theoretically sound solution.

Concerning the numerical results alluded to above, the fact that one is dealing with the subtraction of singular terms makes the chiral extrapolation of the remainder numerically more difficult. In fact, double poles have to be subtracted from the 4-fermion vertex functions [13,37].

3.2. Progress using finite volume schemes

Finite volume schemes allow to compute the scale-dependence of $\Phi_{\rm inter}(\mu)$ explicitly up to large $\mu = \mathrm{O}(100\,\mathrm{GeV})$. One can then verify the onset of perturbative running and use the perturbative β and γ functions (in eq. (7)) only a safe distance beyond that point. In this way one computes $\Phi_{\rm RGI}/\Phi_{\rm inter}(\mu_{\rm m})$, eq. (8), non-perturbatively. Apart from the early studies of renormalized couplings [10,38], only schemes based on the Schrödinger functional [39] have been used so far. The reasons are probably that in this framework

- gauge invariance is explicit,
- on-shell improvement is sufficient,
- observables show good signal/noise ratios, in particular the important running coupling,
- one can use a massless scheme, without extrapolations,
- perturbation theory is relatively easy and
- a-effects are typically quite small.

Still, for new applications one should keep in mind that other finite volume schemes may be useful.

The central objects, which describe the μ-dependence and are computable by MC-methods are the step scaling functions. They give the renormalized quantities $\Phi_{\rm R}$ at scale $\mu = 1/(s\times L)$ as a function of those at scale $\mu = 1/L$. (In this section we use just "R" to denote the intermediate scheme). Picking a complete set of observables, $\Phi_{{\rm R},i}$, the step scaling functions F are

$$\Phi_{{\rm R},i}(\mu/s) = F_i\left(\{\Phi_{{\rm R},j}(\mu)\}\right)\,,\; i,j = 0,\ldots,M\,. \tag{19}$$

A special rôle is played by the renormalized coupling

$$\Phi_{{\rm R},0} \equiv \bar{g}^2(L)\,, \tag{20}$$

which just like in eq. (4) is taken to parameterize the scale $\mu = 1/L$. In fact, in a massless scheme, its step scaling function

$$F_0 \equiv \sigma(u) = \bar{g}^2(sL)\big|_{u=\bar{g}^2(L)} \tag{21}$$

needs no further argument. Another example of a step scaling function is the case of composite operators which mix under renormalization, where eq. (19) is realized as

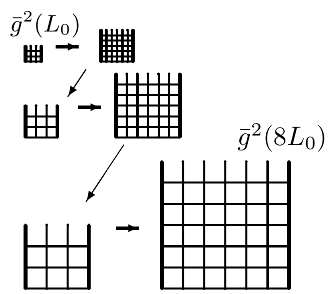
$$\Phi_{{\rm R},i}(\mu/s) = \sum_j \sigma_{ij}(u)\Phi_{{\rm R},j}(\mu)\,, u = \bar{g}^2(L) \tag{22}$$

($\equiv F_i\left(\{\Phi_{\mathrm{R},j}(\mu)\}\right)$). For an example of purely additive renormalization see Sect. 3.3, eq. (37).

Lattice approximants of the step scaling functions are defined at finite resolution a/L via

$$\begin{aligned} \Sigma(u, a/L) &= \bar{g}^2(sL) \qquad \text{at } \bar{g}^2(L) = u \\ \Sigma_{ij}(u, a/L) &= \sum_k Z_{ik}(sL) Z_{kj}^{-1}(L)\,. \end{aligned} \tag{23}$$

As indicated in the illustration for $s = 2$, the r.h.s. of eq. (23) involves quantities on lattices L/a and sL/a but at *the same bare parameters* g_0, m_0 which are fixed by $\bar{g}^2(L) = u$ and the vanishing of the PCAC-mass [11]. Thus the lattice spacing a is kept fixed (and the quark mass zero) while the lattice size L changes by a factor s. In the next step one changes L/a at constant $\bar{g}^2$ (downwards arrows in the graph), thus keeping L fixed instead. This process is iterated to connect $L = L_0$ to $L = L_n = s^n L_0$. To approach the continuum limit, all step scaling functions are computed for several values of a/L at fixed u and extrapolated to the continuum: $\sigma(u) = \lim_{a/L\to 0} \Sigma(u, a/L)$.



Inverting the continuum step scaling functions allows to climb up in energy, $\mu \to s\mu \to s^2\mu \ldots$. Let us illustrate recent progress in implementations of the approach.

1. The basis for all subsequent applications is the running coupling. Following very closely [40,11], it has been studied also for $N_{\mathrm{f}} = 2$. In addition to last year's results [41] also $\Sigma(u, a/L = 1/8)$ is now available [42]. The lattice spacing dependence is very weak and the small effect of dynamical fermions on the (SF-scheme) β-function is seen (Fig. 5). Although some of the simulations were difficult and a modification of the importance sampling was needed to become convinced that the configuration space is sampled properly

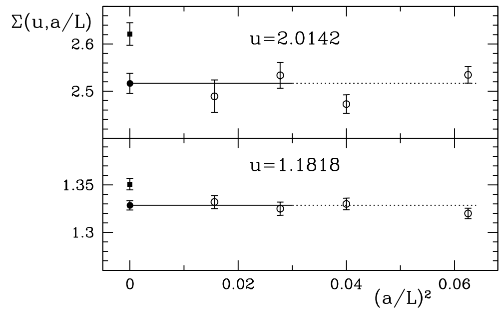


Figure 5. Step scaling function for $N_{\mathrm{f}} = 2$ (circles) compared to the continuum σ for $N_{\mathrm{f}} = 0$ (squares).

in all cases, one finally has a precision result for massless dynamical fermions! Compared to [41] the systematic error due to the $a/L \to 0$ extrapolation is reduced, leading to [42]

$$L_{\max}\Lambda_{\overline{\mathrm{MS}}} = 0.68(7)\,, \text{ where } \bar{g}^2(L_{\max}) = 5\,. \tag{24}$$

2. Defining the quark mass through the renormalized PCAC relation, its scale dependence is given by that of the renormalization constant, Z_{P}, of the flavor non-singlet pseudo-scalar density. In close analogy to [11], its step scaling function, Σ_{P}, has now been computed also for $N_{\mathrm{f}} = 2$ and $L/a = 6, 8$ [1]. A continuum limit requires at least one larger value of L/a.

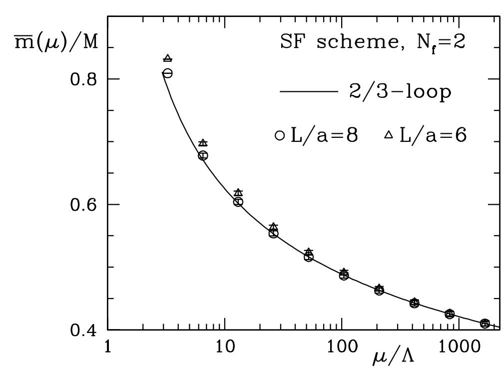


Figure 6. Running of quark masses for $N_{\mathrm{f}} = 2$. For details see [1].

As a preliminary step, the running quark mass in units of the RGI-mass M was evaluated from

$\Sigma_{\rm P}(u,1/6)$ and $\Sigma_{\rm P}(u,1/8)$ separately. The encouraging result is shown in Fig. 6. Although the overall cost of the simulations grows very rapidly with L/a [1], using Hasenbusch's variant of HMC [43] leads to an effort of only about a day on an APEmille crate (peak speed of 64 Gflop/s) for 1% precision in $Z_{\rm P}$ and for $L/a = 16$. Thus, the step $L/a = 12 \to 2L/a = 24$ is not expected to be a problem and the continuum limit is within reach.

Assuming $\Lambda_{\overline{\rm MS}} = {\rm O}(250\,{\rm MeV})$ one finds that the maximum box size, $L_{\rm max}$, is large enough to make contact with the infinite volume physics and extract $\Lambda_{\overline{\rm MS}}/F_\pi$ and $M_{\rm s}/F_\pi$. Still, performing a continuum limit in this step may be a challenge for the future.

3. There are first results for the renormalization of parity-odd 4-fermion operators in the SF-scheme (quenched approximation) [44]. These are of considerable interest since their matrix elements give predictions for

- parity-odd decays when their hadronic matrix elements are evaluated with the standard (improved) Wilson action
- parity-even decays when the standard mass term is replaced by the "twisted" one [45], see also [46]. We refer to this as tmQCD.

The second option is particularly interesting since in this way $B_{\rm K}$ can be computed without any operator mixing.[5]

For the definition of the operators and their renormalization conditions, we refer to [44]. A complete basis has been considered, but so far the detailed analysis has only been done for the operator whose matrix element in tmQCD yields $B_{\rm K}$.

Although in the range $a/L = 1/6 - 1/16$ and for 13 different values of u a lattice spacing dependence of at most 5% has been observed in the step scaling functions, a continuum extrapolation has to be done with care; the operator is not O(a)-improved and linear and quadratic a-effects may compete (see also the case of structure functions [48]). Fortunately there are a number of handles to control the continuum limit. **a)** Five different intermediate schemes have been implemented, which all should yield the same final RGI matrix element $\Phi_{\rm RGI}$. **b)** For the same scheme, different regularizations (e.g. Wilson and improved) have to extrapolate to the same continuum step scaling function [48]. These constraints should be sufficient to obtain accurate final numbers.

4. The renormalization of the static-light axial current in the quenched approximation has been finished, including nice continuum extrapolations of the step scaling functions [3]. The overall Z-factor, mentioned already in the introduction, has then been used to estimate $F_{\rm B_s} = 261(46)\,{\rm MeV} + {\rm O}(1/m_{\rm B_s})$ from the bare matrix elements of [4,49] in the unimproved theory.

3.3. Heavy Quark Effective Theory

As the b-quark mass is usually beyond the cutoff a^{-1} (cf. Fig. 1) it is of great interest to use the $1/m$ expansion, implemented as an effective theory with the (zero-velocity) HQET Lagrangian[50] ($P_+\psi_{\rm h} = \psi_{\rm h}\,,\ \overline{\psi}_{\rm h}P_+ = \overline{\psi}_{\rm h}\,,\ P_\pm = 1 \pm \gamma_0 2$)

$$\mathcal{L}_{\rm HQET} = \mathcal{L}_{\rm h}^{\rm stat} + 1m\mathcal{L}_{\rm h}^{(1)} + {\rm O}(1m^2)\,, \qquad (25)$$

$$\mathcal{L}_{\rm h}^{\rm stat} = \overline{\psi}_{\rm h}(D_0 + m)\psi_{\rm h}\,. \qquad (26)$$

The higher order terms such as

$$\mathcal{L}_{\rm h}^{(1)} = \overline{\psi}_{\rm h}(-12\mathbf{D}^2 - \mathbf{B}\sigma)\psi_{\rm h}\,, \qquad (27)$$

are to be treated as perturbations to the static theory defined by $\mathcal{L}_{\rm h}^{\rm stat}$. Then the expansion is expected to be renormalizable order by order in $1/m$ (similar to chiral perturbation theory). Nevertheless, $\mathcal{L}_{\rm HQET}$ contains operators of different dimension which mix under renormalization with power divergent coefficients $\sim a^{-n}$. Estimating them at a given order k in the coupling constant expansion, one is left with error terms of the form

$$a^{-n} g_0^{2k+2} \xrightarrow{a\to 0} \infty$$

and the continuum limit does not exist.

This is relevant already at the level of the static theory: a linearly divergent ($n = 1$) additive mass renormalization is needed. Unlike the case of relativistic fermions, there is no (chiral) symmetry

[5]With the Wilson action and standard mass term, the relevant 4-fermion operator mixes with operators of wrong chirality. This has been a significant source of uncertainty [47].

to fix this term. Thus, a continuum limit of the b-quark mass in the static approximation requires a *strategy for NP renormalization including the power divergent subtraction.* A viable approach was introduced at last year's conference [51]. The simple idea is again based on the use of finite volume. It is illustrated in Fig. 7, which one should compare to Fig. 1: if we choose the box size L appropriately, we can both

- treat the b-quark as a relativistic fermion, i.e. simulate QCD with $am_{\rm b} \ll 1$ and
- apply HQET quantitatively which requires $Lm_{\rm b} \gg 1$.[6]

Identifying some suitable observables in the

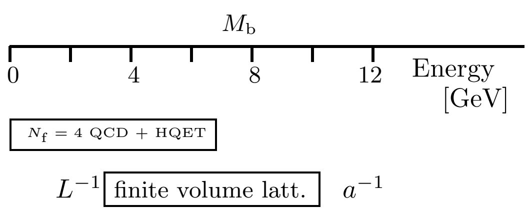


Figure 7. Matching HQET and QCD for $L \approx 0.2$fm

two theories allows to connect the parameters of HQET to those of QCD. This solves the matching problem entirely non-perturbatively (in this case the intermediate step through some RGI's (Fig. 1) is not necessary).

The idea can be implemented in such a way that only quantities which are defined separately in HQET and in QCD and which are independent of the regularization have to be computed. To understand how this works, it is best to consider the following example.

3.3.1. The b-quark mass in the static approximation

The renormalization of the static theory requires (apart from the usual one in the light sector) a (linearly divergent) additive mass renormalization. In other words, when we insert $m = m_{\rm bare}(g_0)$ into the Lagrangian $\mathcal{L}_{\rm h}^{\rm stat}$, eq. (26), with suitably chosen $m_{\rm bare}(g_0)$, then all energies are properly renormalized and have a continuum limit. Due to the simple form of the action, the dependence of the static propagator on $m_{\rm bare}(g_0)$ is explicit and corresponds to a common shift of all energies. It is therefore notationally simpler and common practice to define all observables with $m = 0$ in eq. (26) and keep in mind that renormalized energies are

$$E_{\rm stat,R} = E_{\rm stat} + m_{\rm bare}\,, \text{ with } m = 0. \qquad (28)$$

We now want to use the B-meson mass, $m_{\rm B}$, on the left hand side and relate $m_{\rm bare}$ to the quark mass in the relativistic theory, whose renormalization is under control[11] to obtain the relation between $m_{\rm B}$ and the RGI b-quark mass.

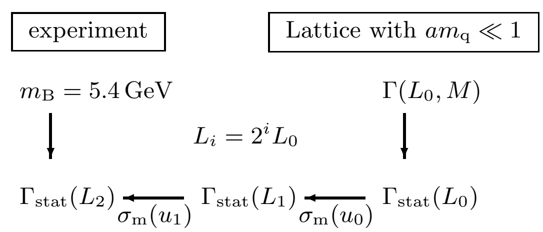
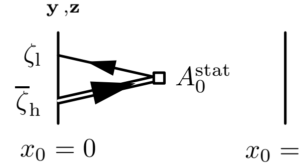


Figure 8. Connecting experimental observables to properly renormalized HQET.

Starting from a static-light correlation function in the SF (see [52] for the precise definition),

$$f_{\rm A}^{\rm stat}(x_0) \propto \sum_{\mathbf{y},\mathbf{z}} \left\langle A_0^{\rm stat}(x)\, \overline{\zeta}_{\rm h}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z})\right\rangle \qquad (29)$$

$=$  (30)

we first define an L-dependent bare energy

$$\Gamma_{\rm stat}(L) = -\tfrac{1}{2}(\partial_0 + \partial_0^*) \ln[f_{\rm A}^{\rm stat}(x_0)] \quad \text{at } x_0 = L/2\,. \qquad (31)$$

The renormalized energy is

$$\Gamma_{\rm stat,R}(L) = \Gamma_{\rm stat}(L) + m_{\rm bare}\,. \qquad (32)$$

The same quantity computed in QCD with finite mass of the heavy quark is denoted by $\Gamma(L, M)$,

[6] These conditions are satisfied for $L \approx 0.2\,$fm, where in comparison to a large volume simulation ($L \approx 2\,$fm) an order of magnitude is gained in the lattice spacing for the same L/a.

where the argument M is conveniently taken as the scheme independent RGI mass [11] of the heavy quark. Matching the two theories at lowest order in $1/M$ is achieved by the condition

$$\Gamma_{\rm stat,R}(L) = \Gamma(L, M_{\rm b})\,. \tag{33}$$

To the same accuracy, the energy of a B-meson, $E_{\rm stat,R}$, equals the physical (experimental) mass, $m_{\rm B}$. Thus we have

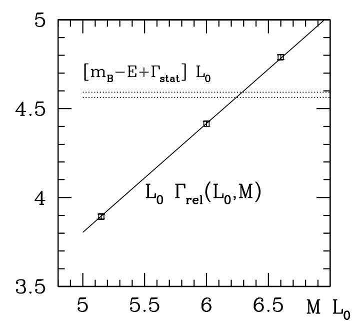


Figure 9. Mass-dependence of $L_0\Gamma(L_0,M)$, in the continuum limit ($N_{\rm f}=0$).

$$\begin{aligned} m_{\rm B} &= E_{\rm stat} + m_{\rm bare} \\ &= E_{\rm stat} + \Gamma(L, M_{\rm b}) - \Gamma_{\rm stat}(L) \\ &= [E_{\rm stat} - \Gamma_{\rm stat}(L_n)] \\ &\quad +[\Gamma_{\rm stat}(L_n) - \Gamma_{\rm stat}(L_0)] + \Gamma(L_0, M_{\rm b})\,, \end{aligned} \tag{34}$$

for arbitrary L_n. Here the matching eq. (33) is performed at $L=L_0$, which has to satisfy

$$L_0 M_{\rm b} \gg 1\,. \tag{35}$$

The two terms in "[ ]"-parenthesis are energy differences in the static theory which do not need renormalization. Their continuum limit can be taken. The entire quark mass dependence is contained in the last term $\Gamma(L_0, M_{\rm b})$, defined in QCD with a relativistic b-quark. For a range of M, the function $\Gamma(L_0, M)$ can again be computed in the continuum limit. The result of a quenched study [51] is shown in Fig. 9.

Finally, all ingredients are present to solve eq. (34) for $M_{\rm b}$. The various steps of the procedure are summarized in Fig. 8.

One point remains to be discussed. Of course the computation of $E_{\rm stat}$ needs a lattice large enough to accommodate a B-meson, say of size $L \geq 1.5\,{\rm fm}$. In this step one can then not afford lattice spacings much smaller than 0.05 fm. At the same lattice spacing (the same bare parameters of the theory), one needs to compute $\Gamma_{\rm stat}(L_n)$ to form the difference $E_{\rm stat} - \Gamma_{\rm stat}(L_n)$. Therefore also L_n may not be too small. In practice, $L_n = 2^n L_0$, $L_0 \approx 0.2\,{\rm fm}$ with $n=2$ is sufficient[51].

To connect $\Gamma_{\rm stat}(L_2)$ to $\Gamma_{\rm stat}(L_0)$ it is furthermore convenient to use

$$\begin{aligned} &[\Gamma_{\rm stat}(L_2) - \Gamma_{\rm stat}(L_0)]\, L_0 = \\ &12\sigma_{\rm m}(u_0) + 14\sigma_{\rm m}(u_1) \quad \text{with } u_i = \bar{g}^2(L_i), \end{aligned} \tag{36}$$

(the Schrödinger functional coupling) and

$$\begin{aligned} \sigma_{\rm m}(u) &= \lim_{a/L\to 0} \Sigma_{\rm m}(u, a/L) \\ \Sigma_{\rm m}(u, a/L) &= 2L\,[\Gamma_{\rm stat}(2L) - \Gamma_{\rm stat}(L)]_{u=\bar{g}^2(L)}\,. \end{aligned} \tag{37}$$

Unnecessary additional scales are avoided by setting the light quark mass to zero everywhere, except for $E_{\rm stat}$. There it is set to the strange quark mass [53]. Correspondingly the experimental spin averaged mass $m_{\rm B} = m_{\rm B_s} = 5405\,{\rm MeV}$ enters.

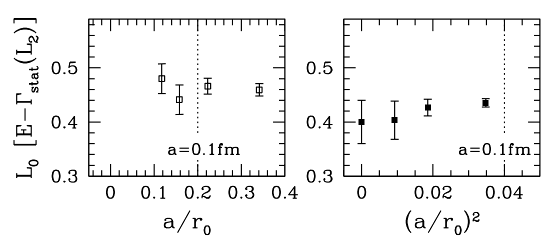


Figure 10. Lattice spacing dependence of the subtracted B-meson energy with O(a) improvement (r.h.s.) and without (l.h.s.).

The method has been tested in the quenched approximation [51]. Except for $E_{\rm stat} - \Gamma_{\rm stat}(L_2)$ all pieces entering eq. (34) have been extrapolated to the continuum. We here just mention few details of this numerical work.

- O(a)-improvement was employed in all steps. Although the improvement coefficient entering the static axial current is known only perturbatively, this did not play a prominent rôle. Its effect on Σ_m is very small.

- The continuum extrapolation of $\Gamma(L_0, M)$ requires to fix L_0 and $z = ML_0$ while varying L_0/a. The proper values of g_0 to keep $\bar{g}^2(L_0) = 2.45$ are known from [40,11] and the bare subtracted quark mass m_q of the heavy quark is obtained from

$$z = ML_0 = L_0\, Z\, Z_M\, m_\mathrm{q}(1 + b_\mathrm{m} a m_\mathrm{q}) \tag{38}$$

The coefficients $Z\,, Z_M\,, b_\mathrm{m}$ are defined in [11,54] and have also been computed non-perturbatively [54]. Nevertheless b_m and Z are needed at relatively small g_0, outside the range considered in [54]. They needed to be extrapolated as a function of g_0, causing additional uncertainties, which can be reduced in future work.

- Values $5.15 \leq z \leq 9$ were used. In this range, $\Gamma(L_0, M)$ turned out to be an almost linear function of z as naively expected. It could be interpolated in z easily (see Fig. 9).

The last piece, $E_\mathrm{stat} - \Gamma_\mathrm{stat}(L_2)$, has now been computed for both the unimproved theory (E_stat from [4]) and with improvement [55] for various lattice spacings. Still the situation, shown in Fig. 10 is not completely satisfactory. The improved results show some residual dependence on the value of the time-separation where E_stat is extracted; this leads to large overall uncertainties. The unimproved results generally are somewhat higher than the improved ones, but do have large errors for $a \leq 0.1\,\mathrm{fm}$. We consider

$$[E_\mathrm{stat} - \Gamma_\mathrm{stat}(L_2)]\, L_0 = 0.40(4)\,, \tag{39}$$

which is shown at $a = 0$, as an estimate of the continuum (quenched) result. Setting the scale through r_0 and using $L_0/r_0 = 1.436/4$ [11], the solution of eq. (34) reads

$$M_\mathrm{b} = 6.96(8)(10)\,\mathrm{GeV}\,, \tag{40}$$

where the larger part of the error originates from the uncertainties of $Z\,, Z_M\,, b_\mathrm{m}$.

3.3.2. Generalizations, Perspectives

The above example shows that the matching in finite volume can be used in practice and for a power-divergent quantity. We now explain briefly how it can be applied more generally.

Again consider multiplicative renormalization, e.g. of the static-light axial current. In terms of a matrix element, $X(L)$, of the bare operator defined in finite volume, the renormalized matrix element is

$$X_\mathrm{R}(L) = Z(g_0, \mu a) X(L)\,, \tag{41}$$

with a renormalization constant $Z(g_0, \mu a)$ defined in some independent way (here μ and L are not related). The desired large volume renormalized matrix element is denoted by

$$\Phi_\mathrm{R} = Z(g_0, \mu a)\Phi\,. \tag{42}$$

In complete analogy to eq. (34) we then have[7]

$$\begin{aligned} \Phi_\mathrm{R} &= \Phi_\mathrm{R}^\mathrm{stat} + \mathrm{O}(1/m) \\ &= \Phi_\mathrm{R}^\mathrm{stat} X_\mathrm{R}^\mathrm{stat}(L_2)\; X_\mathrm{R}^\mathrm{stat}(L_2) X_\mathrm{R}^\mathrm{stat}(L_1)\; X_\mathrm{R}^\mathrm{stat}(L_1) X_\mathrm{R}^\mathrm{stat}(L_0)\;\; X_\mathrm{R}^\mathrm{stat}(L_0) \\ &= \Phi_\mathrm{R}^\mathrm{stat} X_\mathrm{R}^\mathrm{stat}(L_2)\; \sigma_\mathrm{X}(u_1)\; \sigma_\mathrm{X}(u_0)\; X_\mathrm{R}(L_0, M)\,. \end{aligned} \tag{43}$$

The replacement of $X_\mathrm{R}^\mathrm{stat}(L_0)$ by $X_\mathrm{R}(L_0, M)$ is the matching in small volume. The step scaling functions are ($Z(g_0, \mu a)$ cancels out!)

$$\sigma_\mathrm{X}(u) = \lim_{a/L\to 0} \left\{X^\mathrm{stat}(2L) X^\mathrm{stat}(L)\right\}_{\bar{g}^2(L)=u}\,. \tag{44}$$

These formulae are another explicit realization of the strategy Fig. 8 and are easily generalized to include mixing and power subtractions.

All of the above may be applied also beyond the leading order in $1/m$. This opens the exciting perspective to obtain HQET predictions including the first order $1/m$-correction and completely non-perturbative in the QCD coupling. The $1/m$-terms are then treated as insertions in the static propagator (see e.g. [56]). Of course, the matching of all quantities has to be done consistently to order $1/m$.

A potential obstacle on this road to precision predictions for B-physics is the generally rather bad signal-to-noise ratio of static-light correlation functions in large volume [57]. NRQCD suffers little from this problem. Unfortunately in this theory the dimension five operator $\overline{\psi}_\mathrm{h}(-12\mathbf{D}^2)\psi_\mathrm{h}$ is part of the leading order Lagrangian rendering the theory unrenormalizable. It appears impossible to formulate NP-renormalized NRQCD in such a way that a continuum limit can be taken.

[7] In the case of mixing, the step scaling functions become matrices and also the first factor in the last line can easily be given a proper definition.

4. CONCLUSIONS, OPEN PROBLEMS

Over the last years NP renormalization has been developed and applied at a steady pace. Major recent steps have been

- the application of the MOM-scheme with fermions with exact chiral symmetries,
- the renormalization of the static-light axial current
- first steps towards the renormalization of 4-fermion operators in the SF
- the application of the SF renormalization for $N_{\rm f}=2$ and
- the development of a strategy to renormalize HQET non-perturbatively.

The latter, proven to work in an example, should be developed further, in particular to include $1/m$-corrections. Important questions are in how far it is necessary to implement Symanzik improvement and how one can reduce the noise of long distance correlation functions. On the other hand, no particularly small lattice spacings are needed and thus the approach can be used for dynamical fermions.

The computation of $\alpha_{\rm SF}(\mu)$ and the renormalization of composite operators in the SF-scheme have been shown to be applicable in the presence of dynamical fermions without excessive computational requirements. Unfinished parts are mainly due to the connection to the low-energy region of the theory where the usual problems (are the quarks light enough, the volumes large enough?) have to be tackled.

Of course, concerning e.g. $\alpha_{\rm SF}(\mu)$, one needs $N_{\rm f}=3$ dynamical fermions to make better contact with reality. A first step in this direction has been taken by the JLQCD/CP-PACS collaboration who have computed the important improvement coefficient $c_{\rm sw}$ non-perturbatively for $N_{\rm f}=3$, using both the Wilson gauge action and the Iwasaki one [58].

An important open problem is whether the SF may be combined with fermions with exact chiral symmetry. It should be possible to formulate the SF with any kind of regularization, since it is (formally) defined in the continuum. However, no convincing formulation is known as yet. This would be important in order to be able to compute the last factor $Z_{\rm inter}(g_0,\mu_{\rm m}a)$, eq. (8), also in such regularizations without going through yet another quantity defined with periodic boundary conditions or in infinite volume [59].

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REFERENCES

1. F. Knechtli et al., this volume, hep-lat/0209025.
2. S. Aoki et al., Phys. Rev. Lett. 81 (1998) 1778.
3. J. Heitger et al., this volume, hep-lat/0209078.
4. A. Duncan et al., Phys. Rev. D51 (1995) 5101, hep-lat/9407025.
5. G.P. Lepage and P.B. Mackenzie, Phys. Rev. D48 (1993) 2250, hep-lat/9209022.
6. M. Bochicchio et al., Nucl. Phys. B262 (1985) 331; M. Lüscher et al., Nucl. Phys. B491 (1997) 344; P. Hasenfratz et al., hep-lat/0205010.
7. K. Jansen et al., Phys. Lett. B372 (1996) 275.
8. G. Martinelli et al., Nucl. Phys. B445 (1995) 81.
9. P. Hasenfratz, Nucl. Phys. Proc. Suppl. 106 (2002) 159, hep-lat/0111023.
10. M. Lüscher, P. Weisz and U. Wolff, Nucl. Phys. B359 (1991) 221.
11. S. Capitani et al. Nucl. Phys. B544 (1999) 669.
12. S. Sint, Nucl. Phys. Proc. Suppl. 94 (2001) 79; M. Testa, Nucl. Phys. Proc. Suppl. 63 (1998) 38; G.C. Rossi, Nucl. Phys. Proc. Suppl. 53 (1997) 3.

13. J. Reyes et al., this volume, hep-lat/0209131.
14. V. Lubicz, this volume.
15. L. Lellouch, this volume.
16. K. Symanzik, Nucl. Phys. B226 (1983) 187; B. Sheikholeslami and R. Wohlert, Nucl. Phys. B259 (1985) 572; M. Lüscher et al., Nucl. Phys. B478 (1996) 365.
17. M. Lüscher et al., Nucl. Phys. B491 (1997) 323, hep-lat/9609035.
18. D. Becirevic et al., Phys. Rev. D61 (2000) 114507.
19. T. Bhattacharya, R. Gupta and W.J. Lee, Nucl. Phys. Proc. Suppl. 106 (2002) 786.
20. G. Martinelli et al., Nucl. Phys. B611 (2001) 311, hep-lat/0106003.
21. S. Capitani et al., Nucl. Phys. B593 (2001) 183, hep-lat/0007004.
22. S.R. Sharpe, Nucl. Phys. Proc. Suppl. 106 (2002) 817, hep-lat/0110021,
23. L. Giusti, this volume.
24. P.H. Ginsparg and K.G. Wilson, Phys. Rev. D25 (1982) 2649.
25. H. Neuberger, Phys. Lett. B417 (1998) 141, hep-lat/9707022.
26. P. Hasenfratz, V. Laliena and F. Niedermayer, Phys. Lett. B427 (1998) 125.
27. Y. Shamir, Nucl. Phys. B406 (1993) 90; V. Furman and Y. Shamir, Nucl. Phys. B439 (1995) 54.
28. Y. Kikukawa and T. Noguchi, hep-lat/9902022.
29. M. Lüscher, Phys. Lett. B428 (1998) 342.
30. H.B. Nielsen and M. Ninomiya, Phys. Lett. B105 (1981) 219.
31. S. Capitani et al., Phys. Lett. B468 (1999) 150.
32. T. Blum et al., Phys. Rev. D66 (2002) 014504, hep-lat/0102005.
33. L. Giusti, C. Hoelbling and C. Rebbi, Phys. Rev. D64 (2001) 114508, hep-lat/0108007.
34. P.O. Bowman, U.M. Heller and A.G. Williams, Phys. Rev. D66 (2002) 014505.
35. J.R. Cudell, A. Le Yaouanc and C. Pittori, Phys. Lett. B454 (1999) 105.
36. L. Giusti and A. Vladikas, Phys. Lett. B488 (2000) 303, hep-lat/0005026.
37. D. Becirevic et al., JHEP 04 (2002) 025.
38. G. de Divitiis et al., Nucl. Phys. B433 (1995) 390, Nucl. Phys. B437 (1995) 447.
39. M. Lüscher et al., Nucl. Phys. B384 (1992) 168; S. Sint, Nucl. Phys. B421 (1994) 135; S. Sint, Nucl. Phys. B451 (1995) 416.
40. M. Lüscher et al., Nucl. Phys. B413 (1994) 481, hep-lat/9309005.
41. A. Bode et al., Phys. Lett. B **515** (2001) 49.
42. M. Della Morte et al., this volume, hep-lat/0209023.
43. M. Hasenbusch, Phys. Lett. B519 (2001) 177.
44. M. Guagnelli et al., this volume, hep-lat/0209046.
45. R. Frezzotti et al., JHEP 08 (2001) 058; R. Frezzotti et al., JHEP 07 (2001) 048; M. Guagnelli et al., Nucl. Phys. Proc. Suppl. 106 (2002) 320.
46. R. Frezzotti, this volume.
47. L. Lellouch, Nucl. Phys. Proc. Suppl. 94 (2001) 142, hep-lat/0011088.
48. M. Guagnelli, K. Jansen and R. Petronzio, Phys. Lett. B457 (1999) 153.
49. T. Draper and C. McNeile, Nucl. Phys. Proc. Suppl. 34 (1994) 453, hep-lat/9401013.
50. E. Eichten and B. Hill, Phys. Lett. B234 (1990) 511.
51. J. Heitger and R. Sommer, Nucl. Phys. Proc. Suppl. 106 (2002) 358.
52. M. Kurth and R. Sommer, Nucl. Phys. B597 (2001) 488, hep-lat/0007002.
53. J. Garden et al., Nucl. Phys. B571 (2000) 237.
54. M. Guagnelli et al., Nucl. Phys. B595 (2001) 44.
55. H. Molke, S. Dürr, J. Heitger and R. Sommer, in preparation.
56. M. Bochicchio et al., Nucl. Phys. B372 (1992) 403.
57. S. Hashimoto, Phys. Rev. D50 (1994) 4639.
58. S. Aoki, this volume.
59. P. Hernandez et al., JHEP 07 (2001) 018.